ECED 3300
Tutorial 10

Problem 1

A magnetic flux density is given by $\mathbf{B} = \mathbf{a}_y B_0/x$ *Wb/m*^{2}*, where* B_0 *is a constant. A rigid rectangular loop is situated in the* xz*-plane with the corners at the points* (x_0, z_0)*,* $(x_0, z_0 + b)$*,* $(x_0 + a, z_0 + b)$*,* $(x_0 + a, z_0)$*. If the loop is moving with the velocity* $\mathbf{v} = v_0\mathbf{a}_x$*, determine the induced emf.*

Solution

At the time t the corners of the loop will be at the points $(x_0 + vt, z_0)$, $(x_0 + vt, z_0 + b)$, $(x_0 + a + vt, z_0 + b)$, $(x_0 + a + vt, z_0)$. Using the Faraday's law, $\mathcal{E}_{emf} = \oint_C d\mathbf{l} \cdot \mathbf{E} = -\frac{d}{dt}\int_S d\mathbf{S} \cdot \mathbf{B}$. In our case, $d\mathbf{S} = dxdz\mathbf{a}_y$ implying that

$$\int_S d\mathbf{S} \cdot \mathbf{B} = B_0 \int_{z_0}^{z_0+b} dz \int_{x_0+vt}^{x_0+a+vt} \frac{dx}{x} = B_0 b \ln \frac{x_0 + a + vt}{x_0 + vt}.$$

It then follows that

$$\underline{\mathcal{E}_{\mathrm{e}fm} = B_0 bv \left(\frac{1}{x_0 + vt} - \frac{1}{x_0 + a + vt}\right).}$$

Problem 2

Solve the previous problem for a stationary loop in the time-varying magnetic field $\mathbf{B} = \mathbf{a}_y (B_0/x)\cos\omega t$ *Wb/m*^{2}.

Solution

If the loop is at rest, by analogy with the previous example,

$$\begin{aligned}\mathcal{E}_{\mathrm{e}mf} &= -\frac{d}{dt}\int_S d\mathbf{S} \cdot \mathbf{B} = \omega B_0 \sin\omega t \int_{z_0}^{z_0+b} dz \int_{x_0}^{x_0+a} \frac{dx}{x} \\ &= \omega b B_0 \sin\omega t \ln \frac{x_0 + a}{x_0}. \qquad (1)\end{aligned}$$

Thus,

$$\underline{\mathcal{E}_{\mathrm{e}mf} = \omega b B_0 \sin\omega t \ln \frac{x_0 + a}{x_0}.}$$

Problem 3

Assume that the loop in Problem 1 moves with the velocity $\mathbf{v} = v_0\mathbf{a}_x$ in the time-varying field $\mathbf{B} = \mathbf{a}_y(B_0/x)\cos\omega t$ Wb/m^2, find the induced emf.

Solution

In this case, the loop moves and the magnetic flux density changes with time such that

$$\begin{aligned}\mathcal{E}_{emf} &= -B_0\frac{d}{dt}\cos\omega t\int_{z_0}^{z_0+b} dz\int_{x_0+vt}^{x_0+a+vt}\frac{dx}{x} \\ &= -B_0 b\frac{d}{dt}\left(\cos\omega t\ln\frac{x_0+a+vt}{x_0+vt}\right). \qquad (2)\end{aligned}$$

Doing the derivative, we obtain

$$\begin{aligned}\mathcal{E}_{emf} &= B_0 b\left[\omega\sin\omega t\ln\frac{x_0+a+vt}{x_0+vt}+v\cos\omega t\right. \\ &\left.\times\left(\frac{1}{x_0+vt}-\frac{1}{x_0+a+vt}\right)\right]. \qquad (3)\end{aligned}$$

Problem 4

A rod of length l rotates about the z-axis with the angular velocity ω. If $\mathbf{B} = B_0\mathbf{a}_z$, determine the voltage induced in the rod.

Solution

Assume the rod was located along the x-axis at $t = 0$. It follows that at the time t, it makes the angle $\phi = \omega t$ with the x-axis. We apply Faraday's law in the integral form to the sector formed with the x-axis and the position of the rod at the time t.

$$\mathcal{E}_{\rm emf} = -\frac{d}{dt}\int d\mathbf{S}\cdot\mathbf{B} = -\frac{d}{dt}B_0\int_0^{\omega t}d\phi\int_0^l d\rho\,\rho\,(\mathbf{a}_z\cdot\mathbf{a}_z) = -\tfrac{1}{2}B_0 l^2\tfrac{d}{dt}\omega t = -\tfrac{1}{2}B_0 l^2\omega.$$

Thus,

$$V = \mathcal{E}_{\rm emf} = \underline{\tfrac{1}{2}B_0\omega l^2}.$$